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Discrete asymptotic nets and W-congruences in Plücker line geometry

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Abstract

The asymptotic lattices and their transformations are studied within the line geometry approach. It is shown that the discrete asymptotic nets are represented by isotropic congruences in the Plücker quadric. On the basis of the Lelieuvre-type representation of asymptotic lattices and of the discrete analog of the Moutard transformation, it is constructed the discrete analog of the W-congruences, which provide the Darboux–Bäcklund-type transformation of asymptotic lattices. The permutability theorems for the discrete Moutard transformation and for the corresponding transformation of asymptotic lattices are established as well. Moreover, it is proven that the discrete W-congruences are represented by quadrilateral lattices in the quadric of Plücker. These results generalize to a discrete level the classical line geometric approach to asymptotic nets and W-congruences, and incorporate the theory of asymptotic lattices into more general theory of quadrilateral lattices and their reductions.

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1. Introduction

The modern theory of integrable partial differential equations is closely related to the XIX century differential geometry as presented in monographs of Bianchi [2] and Darboux [8]. In that classical period many geometrys studied "interesting" classes of surfaces. A remarkable property of these surfaces (or more appropriate: coordinate systems on surfaces and submanifolds) is that they allow for transformations, which exhibit the so-called *permutability property*. Such transformations called, depending on the context, the Darboux,

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Bianchi, Bäcklund, Laplace, Moutard, Combescure, Lévy, Ribaucour or fundamental transformations of Jonas, can be also described in terms of certain families of lines called line congruences [17,19].

To give an example, the angle between the asymptotic directions on the pseudospherical surfaces in $\mathbb{E}^3$, when written as a function of the asymptotic coordinates satisfies the sine-Gordon equation. From this point of view the study of pseudospherical surfaces is, roughly speaking, equivalent to studying of the sine-Gordon equation and its solutions. The transformations of pseudospherical surfaces introduced by Bianchi and Bäcklund, lead to the celebrated Bäcklund transformations of the sine-Gordon equation.

At the end of XIX century it was also discovered that most of the "interesting" submanifolds are provided by reductions of conjugate nets (see Section 2), and the transformations between such submanifolds are the corresponding reductions of the fundamental (or Jonas) transformations of conjugate nets. It is worth of mentioning that from the point of view of integrable systems the conjugate nets and their iso-conjugate deformations and transformations are described by the so-called multicomponent Kadomtsev–Petviashvilii hierarchy [13].

Apparently, asymptotic nets seem not to be directly related to conjugate nets. However, there exists an approach to asymptotic nets and their transformations (W-congruences) describing them as conjugate nets within the line geometry of Plücker; see Sections 3 and 4 for more details.

In the soliton theory the discrete integrable systems are considered more fundamental then the corresponding differential systems [7,34,36]. Discrete equations include the continuous theory as the result of a limiting procedure, moreover, different limits can give from one discrete equation various differential ones. Furthermore, discrete equations reveal some symmetries lost in the continuous limit.

During last few years the connection between geometry and integrability has been observed also at a discrete level. It turns out that the discrete analogs of pseudospherical surfaces were studied long time ago by Sauer; see [43] and references therein. In connection with the Hirota discrete analog of the sine-Gordon equation [21] these "discrete pseudospherical surfaces" were investigated by Bobenko and Pinkall [4]. In the book of Sauer [43] one can find also other examples of discrete surfaces, or better $\mathbb{Z}^2$ lattices in $\mathbb{R}^3$; in particular, he defined *discrete asymptotic nets* and *discrete conjugate nets* (consult also Sections 2 and 5). These definitions, not only have clear geometric meaning, but also provide the proper, from the point of view of integrability, discretizations of asymptotic and conjugate nets on surfaces.

The importance of discrete conjugate nets in integrability theory was recognized in [10], where it was demonstrated that (the discrete analog of) the Laplace sequence of such lattices provides geometric interpretation of Hirota's discretization of the two-dimensional Toda system [22] — one of the most important equations of the soliton theory and its applications. Soon after that Doliwa and Santini [14] defined and studied the discrete analogs of multidimensional conjugate nets (multidimensional quadrilateral lattices). They also found that the corresponding equations were already known in the literature, being obtained by Bogdanov and Konopelchenko [6] from the $\bar{\partial}$ approach.

The Darboux-type transformations of the quadrilateral lattices have been found by Mañas et al. [37]. The same authors also investigated in detail the geometry of these transformations [16]; in order to do that the theory of discrete congruences has been constructed as well.

In recent literature one can find various examples of integrable discrete geometries (see, for example, [11,12,15] and articles in [3]). It turns out that all the known, up to now, integrable lattices are special cases of asymptotic or quadrilateral lattices. For example, discrete pseudospherical surfaces investigated by Bobenko and Pinkall [4] and discrete affine spheres considered by Bobenko and Schief [5] are asymptotic lattices subjected to additional constraints.

Given a physical system described by integrable partial differential equations, then one of the first steps towards quantizing the model is to find its discrete version preserving the integrability properties (see, for example, [29] and references therein). It turns out that often (see, for example, discussion in [30]) information coming from the quantum model arises naturally as a result of the solution of the classical discrete integrable equations.

Some recent attempts to quantize the theory of gravity use approach of fluctuating geometries (see recent reviews [1,24]) based on the concept of discrete manifolds. However, most of the research in this direction is done by computer simulations, therefore examples of lattice geometries described by integrable equations may be of some help in developing this program. Such integrable lattice geometries could be then studied using powerful tools of the soliton theory, such like the (quantum) inverse spectral transform, algebro-geometric methods of integration, etc.

The connection of asymptotic nets with stationary axially symmetric solutions of the Einstein equations [18] is well known in the literature (see, for example, [35]). Recently, there was discovered by Schief [44] an intriguing link between self-dual Einstein spaces [41] and discrete affine spheres, which form an integrable subcase of discrete asymptotic nets. It is therefore reasonable to study integrability of general asymptotic lattices.

The main results of this paper are contained in Theorems 3 and 4, which incorporate the theory of asymptotic lattices and their transformations into the theory of quadrilateral lattices. These results are direct analogs of the above-mentioned approach to asymptotic nets in terms of conjugate nets in Plücker quadric. The direct proof of integrability of asymptotic lattices, which does not use the theory of quadrilateral lattices, is contained in permutability Theorems 5 and 6.

More detailed description of results is given below:

- Asymptotic lattices are represented in Plücker quadric by isotropic congruences.
- The asymptotic tangents are represented by focal lattices of such congruences.
- The Darboux–Bäcklund transformations of asymptotic lattices are provided by a discrete analog of W-congruences.
- Discrete W-congruences can be constructed from the discrete Moutard transformation via the discrete analog of the Lelieuvre formulas introduced in [27,39].
- The discrete W-congruences are represented in the Plücker quadric by quadrilateral lattices.
- The discussed transformations of asymptotic lattices satisfy the permutability property.

To make our exposition self-contained we first recall necessary results of the theory of conjugate nets and quadrilateral lattices (Section 2) and basic notions of the line geometry (Section 3). Section 4 is intended to motivate our investigations and contains a brief summary of the theory of asymptotic nets and W-congruences. In Section 5, we construct the theory of discrete asymptotic nets within the line geometry of Plücker. Section 6 provides a detailed exposition of the discrete W-congruences. Finally, in Section 7, we state and prove the permutability theorems for the Moutard transformation and for the corresponding W-transformation of asymptotic lattices.

2. Quadrilateral lattices and congruences

In this section, we present basic result from the theory of conjugate nets and congruences [9,17,31], and their discrete generalizations [10,14,16,43]. We give here only definitions necessary to understand results of this paper. In particular, we consider only two-dimensional conjugate nets and lattices.

Definition 1. A coordinate system on a surface in $\mathbb{P}^M$ is called *conjugate net* if tangents to any parametric line transported in the second direction form a developable surface (see Fig. 1).

This geometric characterization can be put into form of the Laplace equation satisfied by homogeneous coordinates $\boldsymbol{y}(v_1, v_2) \in \mathbb{R}^{M+1}$ of the net

$$\partial_1\partial_2\boldsymbol{y} = a\partial_1\boldsymbol{y} + b\partial_2\boldsymbol{y} + c\boldsymbol{y}, \tag{1}$$

here v_1, v_2 are the conjugate parameters, ∂_i denotes the partial derivative with respect to v_i, $i = 1, 2$, and $a(v_1, v_2), b(v_1, v_2), c(v_1, v_2)$ are functions of the conjugate parameters. Given conjugate net on a surface, it defines two new conjugate nets called the Laplace transforms of the old net; the transformations are provided by tangents to the parametric lines (see Fig. 1).

The discrete version of conjugate net on a surface is given by two-dimensional quadrilateral lattice (quadrilateral surface).

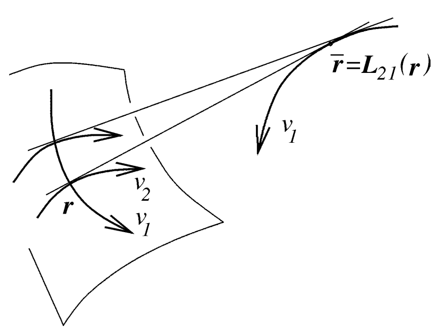

Fig. 1. Conjugate net.

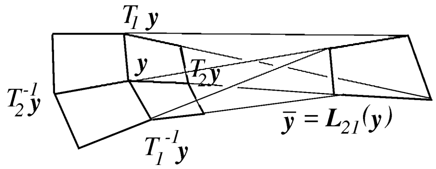

Fig. 2. Quadrilateral surface.

Definition 2. By *quadrilateral surface*, we mean mapping of $\mathbb{Z}^2$ in $\mathbb{P}^M$, such that its elementary quadrilaterals are planar (see Fig. 2).

Remark. Notice that tangents to any parametric discrete curve transported in the second direction form a discrete analog of a developable surface, i.e., one-parameter family of lines tangent to a (discrete) curve.

This geometric characterization implies linear relation between homogeneous coordinates $\boldsymbol{y}(m_1, m_2) \in \mathbb{R}^{M+1}$ of four points of any elementary quadrilateral with vertices $\boldsymbol{y}$, $T_1\boldsymbol{y}$, $T_2\boldsymbol{y}$ and $T_1T_2\boldsymbol{y}$, where T_i denotes shift operator along ith direction of the lattice, $i = 1, 2$. Such a relation can be put into the form of the discrete Laplace equation:

$$\Delta_1\Delta_2\boldsymbol{y} = a\Delta_1\boldsymbol{y} + b\Delta_2\boldsymbol{y} + c\boldsymbol{y}, \tag{2}$$

where $\Delta_i = T_i - 1$, $i = 1, 2$, is the partial difference operator. Intersections of tangent lines define two new quadrilateral surfaces called the Laplace transforms of the old lattice.

Remark. Restriction from $\mathbb{P}^M$ to its affine part, and therefore from homogeneous coordinates to non-homogeneous ones, results in putting $c = 0$ in Eqs. (1) and (2).

The tangents of the lattice are canonical examples of special two-parameter families of straight lines called discrete congruences.

Definition 3. $\mathbb{Z}^2$-parameter family of lines in $\mathbb{P}^M$ is called two-dimensional *discrete congruence* if any two neighboring lines are coplanar.

Remark. Two neighboring tangent lines $\langle[\boldsymbol{y}], T_i[\boldsymbol{y}]\rangle$ and $T_j^{-1}\langle[\boldsymbol{y}], T_i[\boldsymbol{y}]\rangle$, $i \neq j$, of the quadrilateral surface $[\boldsymbol{y}(m_1, m_2)]$ are coplanar and intersect giving $\mathcal{L}_{ij}$ the Laplace transform of the lattice (see Fig. 2).

Definition 4. Intersection points of lines of a discrete congruence with its nearest neighbors in the ith direction form the ith *focal lattice* of the congruence.

One can show that focal lattices of two-dimensional congruences are quadrilateral lattices. The Laplace transformation can be considered as correspondence between focal lattices of a congruence.

Similar notions and results exist in the continuous context.

Definition 5. Two-parameter family of lines in $\mathbb{P}^M$ is called two-dimensional *congruence* if through each line pass two developable surfaces consisting of lines of the family.

One can show that the curves of regression of such developables form two *focal surfaces* tangent to the congruence, moreover, the developables cut the focal surfaces along conjugate nets.

3. Line geometry and the Plücker quadric

The interest of studying families of lines was motivated by the theory of optics, and such mathematicians like Monge, Malus and Hamilton began to create the general theory of rays. However, it was Plücker, who first considered straight lines in $\mathbb{R}^3$ as primary elements; he also found a convenient way to parameterize the space of lines [42]. The geometric interpretation of this parameterization was clarified later by Plücker's pupil Klein [25] and was one of the non-trivial examples in his Erlangen program.

We present in this section basic notions and results of the line geometry, details can be found, for example, in [23,26].

The description of straight lines in $\mathbb{R}^3$ takes more symmetric form if we consider $\mathbb{R}^3$ as the affine part of the projective space $\mathbb{P}^3$ (by the standard embedding $\boldsymbol{y} \mapsto [(\boldsymbol{y}, 1)^{\mathrm{T}}]$), and study straight lines in that space. Given two different points $[\boldsymbol{u}]$, $[\boldsymbol{v}]$ of $\mathbb{P}^3$, the line $\langle[\boldsymbol{u}], [\boldsymbol{v}]\rangle$ passing through them can be represented, up to proportionality factor, by a bi-vector

$$\mathfrak{p} = \boldsymbol{u} \wedge \boldsymbol{v} \in \bigwedge^2(\mathbb{R}^4), \tag{3}$$

changing the reference points of the line results in multiplying the bi-vector by the determinant of the transition matrix between their representatives. The space of straight lines in $\mathbb{P}^3$ can be therefore identified with a subset of $\mathbb{P}(\bigwedge^2(\mathbb{R}^4)) \simeq \mathbb{P}^5$; the necessary and sufficient condition for a non-zero bi-vector $\mathfrak{p}$ in order to represent a straight line is given by the homogeneous equation

$$\mathfrak{p} \wedge \mathfrak{p} = 0, \tag{4}$$

a simple consequence of (3).

If $\boldsymbol{e}_1, \dots, \boldsymbol{e}_4$ is a basis of $\mathbb{R}^4$ then the following bi-vectors:

$$\boldsymbol{e}_{i_1 i_2} = \boldsymbol{e}_{i_1} \wedge \boldsymbol{e}_{i_2}, \quad 1 \leq i_1 < i_2 \leq 4$$

form the corresponding basis of $\bigwedge^2(\mathbb{R}^4)$:

$$\mathfrak{p} = p^{12}\boldsymbol{e}_{12} + p^{13}\boldsymbol{e}_{13} + \cdots + p^{34}\boldsymbol{e}_{34}.$$

Eq. (4) rewritten in the Plücker (or Grassmann–Plücker) coordinates p^{ij} reads

$$p^{12}p^{34} - p^{13}p^{24} + p^{14}p^{23} = 0, \tag{5}$$

and defines in $\mathbb{P}^5$ the so-called Plücker (or Plücker–Klein) quadric $\mathcal{Q}_{\mathrm{P}}$.

Let us present basic subsets of the quadric $\mathcal{Q}_{\mathrm{P}}$ and corresponding configurations of lines in $\mathbb{P}^3$:

1. If two lines intersect then the corresponding bi-vectors $\mathfrak{p}_i$, $i = 1, 2$, satisfy not only equations of the form (4), but also
$$\mathfrak{p}_1 \wedge \mathfrak{p}_2 = 0, \tag{6}$$
i.e., the corresponding points $[\mathfrak{p}_1]$, $[\mathfrak{p}_2]$ of the Plücker quadric are joined by an isotropic (i.e., contained in $\mathcal{Q}_{\mathrm{P}}$) line. Therefore isotropic lines of $\mathbb{P}^5$ correspond to planar pencils of lines in $\mathbb{P}^3$.
2. A conic section of $\mathcal{Q}_{\mathrm{P}}$ by a non-isotropic plane represents the so-called *regulus*, i.e., one family of lines of a ruled quadric in $\mathbb{P}^3$.

4. Asymptotic nets and W-congruences in line geometry

We collect here, for reader's convenience, various results of the theory of asymptotic nets [2,17,19,31], which we consider necessary to understand the methods and goals of next sections where we treat the discrete case.

Definition 6. A coordinate system on a surface in $\mathbb{P}^3$ is called *asymptotic parameterization* if in each point of the surface the osculating planes of the parametric curves coincide with the tangent plane to the surface.

Remark. Through the paper, we consider asymptotic parameterization on a surface in the projective space $\mathbb{P}^3$, but we perform calculations in its affine part $\mathbb{R}^3$.

Given a surface $\boldsymbol{x}(u_1, u_2)$ in $\mathbb{R}^3$ in asymptotic coordinates u_1, u_2 then
$$\partial_1^2 \boldsymbol{x} = a_1 \partial_1 \boldsymbol{x} + b_1 \partial_2 \boldsymbol{x}, \tag{7}$$
$$\partial_2^2 \boldsymbol{x} = a_2 \partial_1 \boldsymbol{x} + b_2 \partial_2 \boldsymbol{x}. \tag{8}$$
As a consequence of the compatibility condition $\partial_1^2 \partial_2^2 \boldsymbol{x} = \partial_2^2 \partial_1^2 \boldsymbol{x}$, we obtain that there exists a function $\phi(u_1, u_2)$ such that
$$a_1 = \partial_1 \phi, \qquad b_2 = \partial_2 \phi.$$
The tangents to the asymptotic lines are represented, in the appropriate gauge, by the bi-vectors
$$\mathfrak{p}_1 = \mathrm{e}^{-\phi} \begin{pmatrix} \boldsymbol{x} \\ 1 \end{pmatrix} \wedge \begin{pmatrix} \partial_1 \boldsymbol{x} \\ 0 \end{pmatrix}, \qquad \mathfrak{p}_2 = \mathrm{e}^{-\phi} \begin{pmatrix} \boldsymbol{x} \\ 1 \end{pmatrix} \wedge \begin{pmatrix} \partial_2 \boldsymbol{x} \\ 0 \end{pmatrix},$$
notice that the line passing through $[\mathfrak{p}_1]$ and $[\mathfrak{p}_2]$ is an isotropic line.

Eqs. (7) and (8) lead to the linear system
$$\partial_1 \mathfrak{p}_1 = b_1 \mathfrak{p}_2, \tag{9}$$
$$\partial_2 \mathfrak{p}_2 = a_2 \mathfrak{p}_1, \tag{10}$$

and, in consequence, to the Laplace equations

$$\partial_1\partial_2\mathfrak{p}_1 = \partial_2(\log b_1)\partial_1\mathfrak{p}_1 + a_2 b_1\mathfrak{p}_1, \qquad \partial_1\partial_2\mathfrak{p}_2 = \partial_1(\log a_2)\partial_2\mathfrak{p}_2 + a_2 b_1\mathfrak{p}_2.$$

The above results can be expressed as follows.

Theorem 1. *A surface in $\mathbb{P}^3$ viewed as the envelope of its tangent planes corresponds to a congruence of isotropic lines of the Plücker quadric $\mathcal{Q}_{\mathrm{P}}$; the focal nets of the congruence represent asymptotic directions of the surface.*

Let us equip $\mathbb{R}^3$ with the scalar product and consider the corresponding cross-product ($\times$). One can show that any asymptotic net $\boldsymbol{x}(u_1, u_2)$ in $\mathbb{R}^3$ can be considered as a solution of the linear system

$$\partial_1\boldsymbol{x} = \partial_1\boldsymbol{N} \times \boldsymbol{N}, \tag{11}$$

$$\partial_2\boldsymbol{x} = \boldsymbol{N} \times \partial_2\boldsymbol{N}, \tag{12}$$

where $\boldsymbol{N}(u_1, u_2)$ is orthogonal to the surface and satisfies equation

$$\partial_1\partial_2\boldsymbol{N} = q\boldsymbol{N} \tag{13}$$

with a function $q(u_1, u_2)$. Eq. (13) was first studied by Moutard [38], and Eqs. (11) and (12) connecting solutions of the Moutard equation with asymptotic nets are known as the Lelieuvre formulas [32].

Remark. It should be mentioned that the Lelieuvre formulas can be settled down within the pure affine (even projective) geometry without referring to additional structures in the ambient space [27].

One can show that $\boldsymbol{N}$ satisfies, in addition to the Moutard equation, the following linear equations:

$$\partial_1^2\boldsymbol{N} = (\partial_1\phi)\partial_1\boldsymbol{N} - b_1\partial_2\boldsymbol{N} + d_1\boldsymbol{N}, \qquad \partial_2^2\boldsymbol{N} = -a_2\partial_1\boldsymbol{N} + (\partial_2\phi)\partial_2\boldsymbol{N} + d_2\boldsymbol{N},$$

where

$$d_1 = \partial_2 b_1 + b_1\partial_2\phi, \qquad d_2 = \partial_1 a_2 + a_2\partial_1\phi,$$

moreover,

$$q = \partial_1\partial_2\phi + b_1 a_2.$$

Given scalar solution $\theta(u_1, u_2)$ of the Moutard equations (13), consider the linear system

$$\partial_1(\theta\hat{\boldsymbol{N}}) = (\partial_1\theta)\boldsymbol{N} - \theta\partial_1\boldsymbol{N}, \tag{14}$$

$$\partial_2(\theta\hat{\boldsymbol{N}}) = -(\partial_2\theta)\boldsymbol{N} + \theta\partial_2\boldsymbol{N}, \tag{15}$$

compatible due to (13). Cross-differentiation of Eqs. (14) and (15) shows that $\hat{\boldsymbol{N}}(u_1, u_2)$ satisfies another Moutard equation

$$\partial_1\partial_2\hat{\boldsymbol{N}} = \hat{q}\hat{\boldsymbol{N}} \tag{16}$$

with the proportionality function $\hat{q}(u_1, u_2)$ given by

$$\hat{q} = \frac{\partial_1\partial_2\hat{\theta}}{\hat{\theta}}, \quad \hat{\theta} = \frac{1}{\theta}.$$

The transition from $\boldsymbol{N}$ to $\hat{\boldsymbol{N}}$ relating solutions of two Moutard equations (13) and (16) is called the Moutard transformation [38].

Simple calculation shows that the surface

$$\hat{\boldsymbol{x}} = \boldsymbol{x} + \hat{\boldsymbol{N}} \times \boldsymbol{N}, \tag{17}$$

can be obtained from $\hat{\boldsymbol{N}}$ via the Lelieuvre formulas. Notice that the straight lines $\langle \boldsymbol{x}, \hat{\boldsymbol{x}}\rangle$ are tangent to both surfaces in corresponding points, i.e., the lines form the so-called Weingarten (or W for short) congruence.

Definition 7. Two-parameter family of straight lines in $\mathbb{P}^3$ tangent to two surfaces in such a way that asymptotic coordinate lines on both surfaces correspond is called *W-congruence*.
There exists another way to find W-congruences tangent to a given asymptotic net $\boldsymbol{x}$. Because $\theta\hat{\boldsymbol{N}} \times \boldsymbol{N}$ is tangent to $\boldsymbol{x}$ therefore it can be decomposed as

$$\theta\hat{\boldsymbol{N}} \times \boldsymbol{N} = A\partial_1\boldsymbol{x} + B\partial_2\boldsymbol{x},$$

the coefficients $A(u_1, u_2)$ and $B(u_1, u_2)$ of the above decomposition define, together with $\boldsymbol{x}(u_1, u_2)$, the W-congruence. It can be shown that the coefficients satisfy the linear system

$$\partial_2 A = -a_2 B, \tag{18}$$

$$\partial_1 B = -b_1 A. \tag{19}$$

Finally, we consider W-congruences in the spirit of Plücker geometry. The bi-vector

$$\mathfrak{q} \propto \begin{pmatrix} \boldsymbol{x} \\ 1 \end{pmatrix} \wedge \begin{pmatrix} \hat{\boldsymbol{x}} \\ 1 \end{pmatrix}$$

represents W-congruence. The bi-vector $\mathfrak{q}$ in the gauge

$$\mathfrak{q} = \theta\, \mathrm{e}^{-\phi} \begin{pmatrix} \boldsymbol{x} \\ 1 \end{pmatrix} \wedge \begin{pmatrix} \hat{\boldsymbol{N}} \times \boldsymbol{N} \\ 0 \end{pmatrix} = A\mathfrak{p}_1 + B\mathfrak{p}_2,$$

satisfies, due to linear systems (9) and (10) and (18) and (19), the Laplace equation

$$\partial_1\partial_2\mathfrak{q} = (\partial_2 \log B)\partial_1\mathfrak{q} + (\partial_1 \log A)\partial_2\mathfrak{q} + [a_2 b_1 - (\partial_1 \log A)(\partial_2 \log B)]\mathfrak{q}.$$

Theorem 2. *W-congruences are represented by conjugate nets in the Plücker quadric* $\mathcal{Q}_{\mathrm{P}}$.

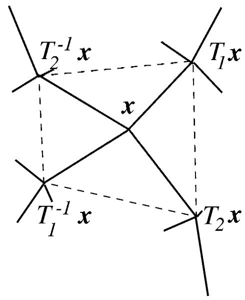

Fig. 3. Asymptotic lattice.

5. Discrete asymptotic nets

Definition 8 (Sauer [43]). An *asymptotic lattice* is a mapping $\boldsymbol{x} : \mathbb{Z}^2 \to \mathbb{R}^3$ such that any point $\boldsymbol{x}$ of the lattice is coplanar with its four nearest neighbors $T_1\boldsymbol{x}$, $T_2\boldsymbol{x}$, $T_1^{-1}\boldsymbol{x}$ and $T_2^{-1}\boldsymbol{x}$ (see Fig. 3).

The plane in Definition 8 can be called the *tangent plane* of the asymptotic lattice in the point $\boldsymbol{x}$.

We can express the asymptotic lattice condition in the form of the linear equations

$$\Delta_1\tilde{\Delta}_1\boldsymbol{x} = a_1\Delta_1\boldsymbol{x} + b_1\Delta_2\boldsymbol{x}, \tag{20}$$

$$\Delta_2\tilde{\Delta}_2\boldsymbol{x} = a_2\Delta_1\boldsymbol{x} + b_2\Delta_2\boldsymbol{x}, \tag{21}$$

where $\tilde{\Delta}_i = 1 - T_i^{-1}$, $i = 1, 2$, is the backward partial difference operator. Eqs. (20) and (21) can be rewritten using the backward tangent vectors as

$$\Delta_1\tilde{\Delta}_1\boldsymbol{x} = \tilde{a}_1\tilde{\Delta}_1\boldsymbol{x} + \tilde{b}_1\tilde{\Delta}_2\boldsymbol{x}, \tag{22}$$

$$\Delta_2\tilde{\Delta}_2\boldsymbol{x} = \tilde{a}_2\tilde{\Delta}_1\boldsymbol{x} + \tilde{b}_2\tilde{\Delta}_2\boldsymbol{x}, \tag{23}$$

here the backward and forward data of the asymptotic lattice are related by the following formulas:

$$\tilde{a}_1 = \frac{1-b_2}{D} - 1, \qquad \tilde{a}_2 = \frac{a_2}{D}, \qquad \tilde{b}_1 = \frac{b_1}{D}, \qquad \tilde{b}_2 = \frac{1-a_1}{D} - 1$$

with

$$D = (1-a_1)(1-b_2) - a_2b_1 = ((1+\tilde{a}_1)(1+\tilde{b}_2) - \tilde{a}_2\tilde{b}_1)^{-1}.$$

The compatibility condition of the linear system (20) and (21) leads, among others, to

$$T_2^{-1}(1-a_1)T_2(1+\tilde{a}_1) = T_1^{-1}(1-b_2)T_1(1+\tilde{b}_2). \tag{24}$$

The backward asymptotic tangent lines can be represented in the line geometry by the bi-vectors

$$\tilde{\mathfrak{p}}_i = \begin{pmatrix} \boldsymbol{x} \\ 1 \end{pmatrix} \wedge \begin{pmatrix} \tilde{\Delta}_i \boldsymbol{x} \\ 0 \end{pmatrix}, \quad i = 1, 2.$$

Using Eqs. (22) and (23) it can be easily shown that

$$T_1\tilde{\mathfrak{p}}_1 = (\tilde{a}_1 + 1)\tilde{\mathfrak{p}}_1 + \tilde{b}_1\tilde{\mathfrak{p}}_2, \tag{25}$$

$$T_2\tilde{\mathfrak{p}}_2 = \tilde{a}_2\tilde{\mathfrak{p}}_1 + (\tilde{b}_2 + 1)\tilde{\mathfrak{p}}_2. \tag{26}$$

Applying to Eq. (25) the shift operator T_2 and using formulas (25) and (26) yields an equivalent form of the discrete Laplace equation

$$T_1T_2\tilde{\mathfrak{p}}_1 = (T_2\tilde{a}_1 + 1)T_2\tilde{\mathfrak{p}}_1 + \frac{T_2\tilde{b}_1}{\tilde{b}_1}(\tilde{b}_2 + 1)T_1\tilde{\mathfrak{p}}_1 - \frac{T_2\tilde{b}_1}{\tilde{b}_1 D}\tilde{\mathfrak{p}}_1,$$

similarly, we get

$$T_1T_2\tilde{\mathfrak{p}}_2 = (T_1\tilde{b}_2 + 1)T_1\tilde{\mathfrak{p}}_2 + \frac{T_1\tilde{a}_2}{\tilde{a}_2}(\tilde{a}_1 + 1)T_2\tilde{\mathfrak{p}}_2 - \frac{T_1\tilde{a}_2}{\tilde{a}_2 D}\tilde{\mathfrak{p}}_2.$$

Notice that the lines $\langle \tilde{\mathfrak{p}}_1, \tilde{\mathfrak{p}}_2 \rangle$ are generators of the Plücker quadric (both asymptotic tangents intersect in $\boldsymbol{x}$) and represent pairs $(\boldsymbol{x}, \pi)$, where π is the tangent plane of the asymptotic lattice at the point $\boldsymbol{x}$. Two neighboring tangent planes π and $T_i^{-1}\pi$, $i = 1, 2$, intersect along the backward tangent line represented by $\tilde{\mathfrak{p}}_i$ (see Fig. 4). We have thus proved the following result.

Theorem 3. *A discrete asymptotic net in* $\mathbb{P}^3$ *viewed as the envelope of its tangent planes corresponds to a congruence of isotropic lines of the Plücker quadric* $\mathcal{Q}_{\mathrm{P}}$; *the focal lattices of the congruence represent asymptotic directions of the lattice*.

Corollary 1. *The lattices in* $\mathcal{Q}_{\mathrm{P}}$ *which represent two families of asymptotic tangents of an asymptotic lattice are Laplace transforms of each other*.

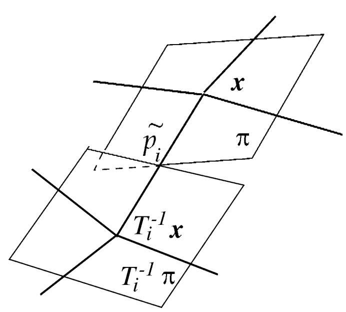

Fig. 4. Asymptotic directions as focal lattices of the isotropic congruence.

Similarly, like in the continuous case there exist the discrete analog of the Lelieuvre representation and the discrete analog of the Moutard equation; for details, see [27,39]. It can be shown that

$$\Delta_1 \boldsymbol{x} = \Delta_1 \boldsymbol{N} \times \boldsymbol{N}, \tag{27}$$

$$\Delta_2 \boldsymbol{x} = \boldsymbol{N} \times \Delta_2 \boldsymbol{N}, \tag{28}$$

where the vector $\boldsymbol{N}$, orthogonal to the tangent plane of the lattice, satisfies the discrete Moutard equation (see also [40])

$$T_1 T_2 \boldsymbol{N} + \boldsymbol{N} = Q(T_1 \boldsymbol{N} + T_2 \boldsymbol{N}),$$

whose equivalent form is

$$\Delta_1 \Delta_2 \boldsymbol{N} = (Q - 1)(\Delta_1 \boldsymbol{N} + \Delta_2 \boldsymbol{N} + 2\boldsymbol{N}). \tag{29}$$

We would like to add some new ingredients to the connection of the Lelieuvre representation of the asymptotic lattices and the linear system (20) and (21). The normal vector satisfies equations

$$\Delta_1 \tilde{\Delta}_1 \boldsymbol{N} = a_1 \Delta_1 \boldsymbol{N} - b_1 \Delta_2 \boldsymbol{N} + d_1 \boldsymbol{N}, \tag{30}$$

$$\Delta_2 \tilde{\Delta}_2 \boldsymbol{N} = -a_2 \Delta_1 \boldsymbol{N} + b_2 \Delta_2 \boldsymbol{N} + d_2 \boldsymbol{N}. \tag{31}$$

The compatibility condition of the system (30) and (31) with the Moutard equation (29) give

$$(1 - b_2) T_1 (1 + \tilde{b}_2) = Q(T_2^{-1} Q), \tag{32}$$

$$(1 - a_1) T_2 (1 + \tilde{a}_1) = Q(T_1^{-1} Q). \tag{33}$$

Combining Eqs. (32) and (33) with (24) yields the following identity:

$$\frac{T_1 T_2 \tilde{a}_1 + 1}{(T_1 Q)(T_2 D)(T_2 \tilde{a}_1 + 1)} = \frac{T_1 T_2 \tilde{b}_2 + 1}{(T_2 Q)(T_1 D)(T_1 \tilde{b}_2 + 1)} = F, \tag{34}$$

which will be used in the next section.

6. Discrete W-congruences

Similarly, like in the continuous case, given solution $\Theta(n_1, n_2)$ of the discrete Moutard equation (29), one can define the (discrete analog of the) Moutard transformation [39] (see also [40]) by solving the linear system

$$\Delta_1 (\Theta \hat{\boldsymbol{N}}) = (\Delta_1 \Theta) \boldsymbol{N} - \Theta \Delta_1 \boldsymbol{N}, \tag{35}$$

$$\Delta_2 (\Theta \hat{\boldsymbol{N}}) = -(\Delta_2 \Theta) \boldsymbol{N} + \Theta \Delta_2 \boldsymbol{N}, \tag{36}$$

which yields

$$T_1T_2\hat{\boldsymbol{N}} + \hat{\boldsymbol{N}} = \hat{Q}(T_1\hat{\boldsymbol{N}} + T_2\hat{\boldsymbol{N}})$$

with new proportionality factor

$$\hat{Q} = \frac{T_1T_2\hat{\Theta} + \hat{\Theta}}{T_1\hat{\Theta} + T_2\hat{\Theta}}, \quad \hat{\Theta} = \frac{1}{\Theta}.$$

Let us define the following lattice:

$$\hat{\boldsymbol{x}} = \boldsymbol{x} + \hat{\boldsymbol{N}} \times \boldsymbol{N}, \tag{37}$$

a simple calculation shows that formula (37) gives new asymptotic lattice with the normal vector $\hat{\boldsymbol{N}}$ entering into the Lelieuvre formulas.

The line $\langle \boldsymbol{x}, \hat{\boldsymbol{x}} \rangle$ is tangent to both lattices, therefore, we have

$$\Theta\hat{\boldsymbol{N}} \times \boldsymbol{N} = A\Delta_1\boldsymbol{x} + B\Delta_2\boldsymbol{x} = \tilde{A}\tilde{\Delta}_1\boldsymbol{x} + \tilde{B}\tilde{\Delta}_2\boldsymbol{x}, \tag{38}$$

where

$$\tilde{A} = A(\tilde{a}_1 + 1) + B\tilde{a}_2, \tag{39}$$

$$\tilde{B} = A\tilde{b}_1 + B(\tilde{b}_2 + 1). \tag{40}$$

Notice that the two-parameter family of lines $\langle \boldsymbol{x}, \hat{\boldsymbol{x}} \rangle$ has analogous properties of that of the *W-congruence* from continuous case.

Definition 9. By a *discrete W-congruence*, we mean two-parameter family of straight lines connecting two asymptotic lattices in such a way that the lines are tangent to the lattices in corresponding points.

We have shown how the get discrete W-congruences from the Moutard transformations. It turns out that any discrete W-congruence can be obtained in this way.

Proposition 1. *Given discrete W-congruence connecting $\boldsymbol{x}$ and $\hat{\boldsymbol{x}}$, then the normal vectors $\boldsymbol{N}$ and $\hat{\boldsymbol{N}}$ which define $\boldsymbol{x}$ and $\hat{\boldsymbol{x}}$ via the Lelieuvre formulas, are related by a Moutard transformation.*

$$\begin{tikzcd} \boldsymbol{x} \arrow[r, "W-congruence"] \arrow[d] & \widehat{\boldsymbol{x}} \arrow[d] \\ \boldsymbol{N} \arrow[r, "Moutard\ transf."] & \widehat{\boldsymbol{N}} \end{tikzcd}$$

Proof. From Definition 9 it follows that $\hat{\boldsymbol{x}}$ must be of the form

$$\hat{\boldsymbol{x}} = \boldsymbol{x} + \psi\hat{\boldsymbol{N}} \times \boldsymbol{N}. \tag{41}$$

We first show that without loss of generality one can put the proportionality function $\psi(n_1, n_2)$ equal to 1. Applying the partial difference operator Δ_1 to Eq. (41) and using the first part (27) of the discrete Lelieuvre formulas, we get

$$T_1\hat{\boldsymbol{N}} \times \boldsymbol{N} = T_1\boldsymbol{N} \times \boldsymbol{N} + (T_1\psi)T_1\hat{\boldsymbol{N}} \times T_1\boldsymbol{N} - \psi\hat{\boldsymbol{N}} \times \boldsymbol{N}. \tag{42}$$

The scalar products with $T_1\hat{\boldsymbol{N}}$ and with $\boldsymbol{N}$ give

$$(T_1\boldsymbol{N} \times \boldsymbol{N}) \cdot T_1\hat{\boldsymbol{N}} = \psi(\hat{\boldsymbol{N}} \times \boldsymbol{N}) \cdot T_1\hat{\boldsymbol{N}}, \qquad (T_1\hat{\boldsymbol{N}} \times \hat{\boldsymbol{N}}) \cdot \boldsymbol{N} = T_1\psi(T_1\boldsymbol{N} \times \boldsymbol{N}) \cdot \boldsymbol{N},$$

which after simple manipulation gives

$$(T_1\psi)\psi = 1, \tag{43}$$

similarly, we have

$$(T_2\psi)\psi = 1. \tag{44}$$

Notice that due to Eqs. (43) and (44) the normal vector $\hat{\boldsymbol{N}}' = \psi\hat{\boldsymbol{N}}$ defines the same lattice $\hat{\boldsymbol{x}}$, which shows that in formula (41), we can put $\psi \equiv 1$.

After such a change, formula (42) can be rewritten in the form

$$(T_1\boldsymbol{N} - \hat{\boldsymbol{N}}) \times (\boldsymbol{N} - T_1\hat{\boldsymbol{N}}) = 0,$$

which yields

$$T_1\boldsymbol{N} - \hat{\boldsymbol{N}} = \lambda(\boldsymbol{N} - T_1\hat{\boldsymbol{N}}). \tag{45}$$

Similarly, we obtain

$$T_2\boldsymbol{N} + \hat{\boldsymbol{N}} = \mu(\boldsymbol{N} + T_2\hat{\boldsymbol{N}}). \tag{46}$$

Formulas (45) and (46) give together with the Moutard equations satisfied by $\boldsymbol{N}$ and $\hat{\boldsymbol{N}}$, the following equations:

$$\lambda Q = (T_2\lambda)\hat{Q}, \qquad \mu Q = (T_1\mu)\hat{Q},$$
$$\mu Q - 1 = (T_2\lambda)(\mu - \hat{Q}), \qquad \lambda Q - 1 = (T_1\mu)(\lambda - \hat{Q}).$$

This gives

$$(T_2\lambda)\mu = (T_1\mu)\lambda,$$

which implies that

$$\lambda = \frac{T_1\Theta}{\Theta}, \qquad \mu = \frac{T_2\Theta}{\Theta}, \tag{47}$$

moreover, Θ satisfies the Moutard equation of $\boldsymbol{N}$. Finally, Eqs. (45) and (46) with λ and μ given by (47) can be put in the form of the Moutard transformation (35) and (36). $\square$

Eq. (38) imply

$$\Theta\hat{\boldsymbol{N}} = \tilde{A}\tilde{\Delta}_1\boldsymbol{N} - \tilde{B}\tilde{\Delta}_2\boldsymbol{N} + \tilde{C}\boldsymbol{N}. \tag{48}$$

The compatibility condition of Eq. (48) and the Moutard transformation (35) and (36) gives, among others,

$$T_1\left(\frac{\tilde{B}}{\tilde{b}_2+1}\right) = \frac{B}{Q}, \tag{49}$$

$$T_2\left(\frac{\tilde{A}}{\tilde{a}_1+1}\right) = \frac{A}{Q}. \tag{50}$$

The following result is the generalization of Theorem 2 to the discrete case.

Theorem 4. *Discrete W-congruences are represented by two-dimensional quadrilateral lattices in the Plücker quadric* $\mathcal{Q}_{\mathrm{P}}$.

Proof. Lines of the W-congruence are represented by bi-vectors

$$\mathfrak{q} = \begin{pmatrix} \boldsymbol{x} \\ 1 \end{pmatrix} \wedge \begin{pmatrix} \Theta\boldsymbol{N}\times\boldsymbol{N} \\ 0 \end{pmatrix} = \tilde{A}\tilde{\mathfrak{p}}_1 + \tilde{B}\tilde{\mathfrak{p}}_2.$$

We will show that $\mathfrak{q}$ satisfies the Laplace equation.

Because of (25) and (26), we have

$$T_1\mathfrak{q} = T_1\tilde{A}[(\tilde{a}_1+1)\tilde{\mathfrak{p}}_1 + \tilde{b}_1\tilde{\mathfrak{p}}_2] + (T_1\tilde{B})T_1\tilde{\mathfrak{p}}_2,$$
$$T_2\mathfrak{q} = T_2\tilde{B}[(\tilde{b}_2+1)\tilde{\mathfrak{p}}_2 + \tilde{a}_2\tilde{\mathfrak{p}}_1] + (T_2\tilde{A})T_2\tilde{\mathfrak{p}}_1,$$

and therefore

$$T_1T_2\mathfrak{q} = \frac{T_1T_2\tilde{A}}{T_2\tilde{A}}T_2(\tilde{a}_1+1)T_2\mathfrak{q} + \frac{T_1T_2\tilde{B}}{T_1\tilde{B}}T_1(\tilde{b}_2+1)T_1\mathfrak{q} + U\tilde{\mathfrak{p}}_1 + V\tilde{\mathfrak{p}}_2, \tag{51}$$

where

$$U = \tilde{a}_2\frac{T_1T_2\tilde{A}}{T_2\tilde{A}}T_2(\tilde{A}\tilde{b}_1 - \tilde{B}(\tilde{a}_1+1)) + (\tilde{a}_1+1)\frac{T_1T_2\tilde{B}}{T_2\tilde{B}}T_1(\tilde{B}\tilde{a}_2 - \tilde{A}(\tilde{b}_2+1)),$$
$$V = (\tilde{b}_2+1)\frac{T_1T_2\tilde{A}}{T_2\tilde{A}}T_2(\tilde{A}\tilde{b}_1 - \tilde{B}(\tilde{a}_1+1)) + \tilde{b}_1\frac{T_1T_2\tilde{B}}{T_2\tilde{B}}T_1(\tilde{B}\tilde{a}_2 - \tilde{A}(\tilde{b}_2+1)).$$

Using Eqs. (39) and (40), we get

$$U = -\tilde{a}_2\frac{T_1T_2\tilde{A}}{T_2\tilde{A}}T_2\left(\frac{B}{D}\right) - (\tilde{a}_1+1)\frac{T_1T_2\tilde{B}}{T_2\tilde{B}}T_1\left(\frac{A}{D}\right),$$

which, due to Eqs. (49) and (50), can be transformed to

$$U = -Q(T_1A)(T_2B)\left(\frac{\tilde{a}_2(T_1T_2\tilde{a}_1+1)}{A(T_1Q)(T_2D)(T_2\tilde{a}_1+1)} + \frac{(\tilde{a}_1+1)(T_1T_2\tilde{b}_2+1)}{B(T_2Q)(T_1D)(T_1\tilde{b}_2+1)}\right).$$

Identity (34) gives together with Eqs. (39) and (40) that

$$U = -\tilde{A}QF\frac{(T_1A)(T_2B)}{AB},$$

similarly,

$$V = -\tilde{B}QF\frac{(T_1A)(T_2B)}{AB},$$

which yields

$$U\tilde{\mathfrak{p}}_1 + V\tilde{\mathfrak{p}}_2 = -QF\frac{(T_1A)(T_2B)}{AB}\mathfrak{q}. \tag{52}$$

Inserting (52) into Eq. (51) leads to conclusion that the bi-vector $\mathfrak{q}$ satisfies the Laplace equation. □

From the interpretation of W-congruences as quadrilateral lattices in $\mathcal{Q}_{\mathrm{P}}$, we infer the following property (see final remarks of Section 3).

Corollary 2. *Four neighboring lines of a W-congruence are generators of a ruled quadric in* $\mathbb{P}^3$.

We would like to stress that the discrete W-congruences are not discrete congruences in the sense of Definition 3. In order to explain this terminological confusion we would like to make a few historical comments. At the beginning of the line geometry, by a congruence it was meant any two-parameter family of straight lines in $\mathbb{R}^3$. It turns out that in $\mathbb{R}^3$ such family has, in general, two focal surfaces. However, in more dimensional ambient space two-parameter families of lines do not have, in general, focal surfaces.

From the point of view of transformations of surfaces it was necessary, therefore, to put some restrictions on the initial definition, and we read in [17, p. 11]: "we call a *congruence* in n-space a two-parameter family of lines such that through each line pass two developable surfaces of the family." Going further into multiparameter families of lines and into discrete domain, in order to keep the basic property of congruences they have been defined [16] in such a way that they have focal lattices; this requirement leads to Definition 3. In continuous case W-congruences have focal surfaces, but this is, as we mentioned above, typical property of two-parameter families of lines in $\mathbb{R}^3$. In our opinion, this terminological confusion suggests that it is more convenient to consider discrete W-congruences as *quadrilateral lattices in the line space*.

7. Permutability theorems

In this section, we consider superposition of the Moutard transformations and the corresponding superpositions of W-transformations of asymptotic lattices. We prove also the permutability theorems for both transformations.

Let $\Theta^1(n_1, n_2)$ and $\Theta^2(n_1, n_2)$ be two solutions of the Moutard equation of the lattice $\boldsymbol{N}(n_1, n_2)$, i.e.,

$$T_1 T_2 \begin{pmatrix} \boldsymbol{N} \\ \Theta^1 \\ \Theta^2 \end{pmatrix} + \begin{pmatrix} \boldsymbol{N} \\ \Theta^1 \\ \Theta^2 \end{pmatrix} = Q \left[T_1 \begin{pmatrix} \boldsymbol{N} \\ \Theta^1 \\ \Theta^2 \end{pmatrix} + T_2 \begin{pmatrix} \boldsymbol{N} \\ \Theta^1 \\ \Theta^2 \end{pmatrix} \right]. \tag{53}$$

We use Θ^1 to define the first Moutard transformation $\boldsymbol{N}_1$ of the lattice $\boldsymbol{N}$ and the corresponding transformation Θ^2_1 of Θ^2 via Eqs. (35) and (36):

$$\Delta_1 \left[\Theta^1 \begin{pmatrix} \boldsymbol{N}_1 \\ \Theta^2_1 \end{pmatrix} \right] = (\Delta_1 \Theta^1) \begin{pmatrix} \boldsymbol{N} \\ \Theta^2 \end{pmatrix} - \Theta^1 \Delta_1 \begin{pmatrix} \boldsymbol{N} \\ \Theta^2 \end{pmatrix}, \tag{54}$$

$$\Delta_2 \left[\Theta^1 \begin{pmatrix} \boldsymbol{N}_1 \\ \Theta^2_1 \end{pmatrix} \right] = -(\Delta_2 \Theta^1) \begin{pmatrix} \boldsymbol{N} \\ \Theta^2 \end{pmatrix} + \Theta^1 \Delta_2 \begin{pmatrix} \boldsymbol{N} \\ \Theta^2 \end{pmatrix}, \tag{55}$$

which implies that both $\boldsymbol{N}_1$ and Θ^2_1 satisfy the same Moutard equation

$$T_1 T_2 \begin{pmatrix} \boldsymbol{N}_1 \\ \Theta^2_1 \end{pmatrix} + \begin{pmatrix} \boldsymbol{N}_1 \\ \Theta^2_1 \end{pmatrix} = Q_1 \left[T_1 \begin{pmatrix} \boldsymbol{N}_1 \\ \Theta^2_1 \end{pmatrix} + T_2 \begin{pmatrix} \boldsymbol{N}_1 \\ \Theta^2_1 \end{pmatrix} \right],$$

where

$$Q_1 = \frac{T_1 T_2 \hat{\Theta}^1 + \hat{\Theta}^1}{T_1 \hat{\Theta}^1 + T_2 \hat{\Theta}^1}, \quad \hat{\Theta}^1 = \frac{1}{\Theta^1}.$$

Similarly, we use Θ^2 to define the second Moutard transformation $\boldsymbol{N}_2$ of the lattice $\boldsymbol{N}$ and the corresponding transformation Θ^1_2 of Θ^1:

$$\Delta_1 \left[\Theta^2 \begin{pmatrix} \boldsymbol{N}_2 \\ \Theta^1_2 \end{pmatrix} \right] = -(\Delta_1 \Theta^2) \begin{pmatrix} \boldsymbol{N} \\ \Theta^1 \end{pmatrix} + \Theta^2 \Delta_1 \begin{pmatrix} \boldsymbol{N} \\ \Theta^1 \end{pmatrix}, \tag{56}$$

$$\Delta_2 \left[\Theta^2 \begin{pmatrix} \boldsymbol{N}_2 \\ \Theta^1_2 \end{pmatrix} \right] = (\Delta_2 \Theta^2) \begin{pmatrix} \boldsymbol{N} \\ \Theta^1 \end{pmatrix} - \Theta^2 \Delta_2 \begin{pmatrix} \boldsymbol{N} \\ \Theta^1 \end{pmatrix}. \tag{57}$$

Notice the modification of signs in the transformation formulas, which, however, do not change the fact that both $\boldsymbol{N}_2$ and Θ^1_2 satisfy the same Moutard equation

$$T_1 T_2 \begin{pmatrix} \boldsymbol{N}_2 \\ \Theta^1_2 \end{pmatrix} + \begin{pmatrix} \boldsymbol{N}_2 \\ \Theta^1_2 \end{pmatrix} = Q_2 \left[T_1 \begin{pmatrix} \boldsymbol{N}_2 \\ \Theta^1_2 \end{pmatrix} + T_2 \begin{pmatrix} \boldsymbol{N}_2 \\ \Theta^1_2 \end{pmatrix} \right],$$

where

$$Q_2 = \frac{T_1 T_2 \hat{\Theta}^2 + \hat{\Theta}^2}{T_1 \hat{\Theta}^2 + T_2 \hat{\Theta}^2}, \quad \hat{\Theta}^2 = \frac{1}{\Theta^2}.$$

Eqs. (54)–(57) imply that both products $\Theta^1 \Theta^2_1$ and $\Theta^2 \Theta^1_2$ are defined up to additive constants. Moreover, since

$$\Delta_1(\Theta^1 \Theta^2_1) = \Delta_1(\Theta^1)\Theta^2 - \Theta^1 \Delta_1 \Theta^2 = \Delta_1(\Theta^2 \Theta^1_2),$$
$$\Delta_2(\Theta^1 \Theta^2_1) = -\Delta_2(\Theta^1)\Theta^2 + \Theta^1 \Delta_2 \Theta^2 = \Delta_2(\Theta^2 \Theta^1_2),$$

then one of these constants can be fixed in such a way that

$$\Theta^1\Theta^2_1 = \Theta^2\Theta^1_2 = \Xi^{12} \tag{58}$$

holds.

The following result states that there exist lattices being simultaneous Moutard transformations of $\boldsymbol{N}_1$ and $\boldsymbol{N}_2$, what can be illustrated by the diagram

$$\begin{tikzcd} \boldsymbol{N} \arrow[r, "\Theta^1"] \arrow[d, "\Theta^2"'] & \boldsymbol{N}_1 \arrow[d, "\Theta^2_1"] \\ \boldsymbol{N}_2 \arrow[r, "\Theta^1_2"] & \boldsymbol{N}_{12}. \end{tikzcd}$$

Theorem 5 (Permutability of the Moutard transformations). *Let Θ^1, Θ^2 be solutions of the discrete Moutard equation of the lattice $\boldsymbol{N}$, and let $\boldsymbol{N}_1$, $\boldsymbol{N}_2$ be the corresponding two (discrete) Moutard transformations of $\boldsymbol{N}$. Then the functions Θ^1_2 and Θ^2_1, given by Eqs.* (54)–(58), *provide by the formula*

$$\boldsymbol{N}_{12} + \boldsymbol{N} = \frac{\Theta^1\Theta^2}{\Xi^{12}}(\boldsymbol{N}_1 + \boldsymbol{N}_2), \tag{59}$$

one-parameter family (because of the free integration constant in Ξ^{12}) of the Moutard transformations of the lattice $\boldsymbol{N}_1$ (by means of the function Θ^2_1) which are simultaneously the Moutard transformation of the lattice $\boldsymbol{N}_2$ (by means of the function Θ^1_2).

Proof. It is enough to verify directly that the lattice $\boldsymbol{N}_{12} = \boldsymbol{N}_{21}$ given by (59) is a solution of equations

$$\Delta_1(\Theta^1_2\boldsymbol{N}_{21}) = (\Delta_1\Theta^1_2)\boldsymbol{N}_2 - \Theta^1_2\Delta_1\boldsymbol{N}_2, \quad \Delta_2(\Theta^1_2\boldsymbol{N}_{21}) = -(\Delta_2\Theta^1_2)\boldsymbol{N}_2 + \Theta^1_2\Delta_2\boldsymbol{N}_2,$$

which define the Moutard transformations of $\boldsymbol{N}_2$ by means of Θ^1_2, and that it is also a solution of equations

$$\Delta_1(\Theta^2_1\boldsymbol{N}_{12}) = -(\Delta_1\Theta^2_1)\boldsymbol{N}_1 + \Theta^2_1\Delta_1\boldsymbol{N}_1, \quad \Delta_2(\Theta^2_1\boldsymbol{N}_{12}) = (\Delta_2\Theta^2_1)\boldsymbol{N}_1 - \Theta^2_1\Delta_2\boldsymbol{N}_1,$$

which define the Moutard transformations of $\boldsymbol{N}_1$ by means of Θ^2_1. □

Remark. Notice that the superposition formula (59) itself is of the form of the discrete Moutard equation. This is a manifestation of the frequently observed relation between discrete integrable systems and their Darboux-type transformations [4,16,20,28,33,40]. To obtain such a form of the superposition formula it was the reason of modification of signs in the Moutard transformation (56) and (57).

The corresponding theorem (the discrete analog of the classical Bianchi permutability theorem [2,17]) about permutability of the W-transformations of asymptotic lattices reads as follows.

Theorem 6 (Permutability of the W-transformations). *If $\boldsymbol{x}$ and $\boldsymbol{x}_1$ are asymptotic lattices related by a W-congruence and $\boldsymbol{x}$ and $\boldsymbol{x}_2$ are related by a second W-congruence then there can be found one-parameter family of asymptotic lattices given in notation of Theorem* 5 *by*

$$\boldsymbol{x}_{12} = \boldsymbol{x}_{21} = \boldsymbol{x} + \frac{\Theta^1\Theta^2}{\Xi^{12}}\boldsymbol{N}_1 \times \boldsymbol{N}_2, \tag{60}$$

such that $\boldsymbol{x}_1$ and $\boldsymbol{x}_{12}$ are related by a W-congruence, and likewise $\boldsymbol{x}_2$ and $\boldsymbol{x}_{12}$.

Proof. Due to Proposition 1, the lattices $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ can be given by

$$\boldsymbol{x}_1 = \boldsymbol{x} + \boldsymbol{N}_1 \times \boldsymbol{N}, \tag{61}$$

$$\boldsymbol{x}_2 = \boldsymbol{x} - \boldsymbol{N}_2 \times \boldsymbol{N}, \tag{62}$$

where $\boldsymbol{N}$, $\boldsymbol{N}_1$ and $\boldsymbol{N}_2$ are like in (54)–(57); notice the change of sign in (62) induced by the change of sign in (56) and (57). Transforming lattice $\boldsymbol{x}_1$ by (62) by means of $\boldsymbol{N}_{12}$ and applying (59) one obtains (60), likewise transforming lattice $\boldsymbol{x}_2$ by formula (61) by means of $\boldsymbol{N}_{12}$. □

8. Conclusion and remarks

The main result of this paper consists in showing that the theory of asymptotic lattices and their transformations given by W-congruences forms a part of the theory of quadrilateral lattices. The discrete W-congruences can be considered as quadrilateral lattices in the Plücker quadric, therefore they provide non-trivial examples of quadrilateral lattices subjected to quadratic constraints, whose general theory was constructed in [12]. We demonstrated also the permutability property of the corresponding W-transformations of asymptotic lattices, thus proving directly their integrability.

Our result is the next step in realization of the general program of classification of integrable geometries as reductions of quadrilateral lattices. Such reductions come usually from additional structures in the projective ambient space (the close analogy to the Erlangen program of Klein), and/or from inner symmetries of the lattice itself (see also examples in [15]); in our case the basic underlying geometry is the line geometry of Plücker.

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